NONISOTHERMAL FLOW OF A VISCOUS INCOMPRESSIBLE FLUID IN A TWO-DIMENSIONAL DIFFUSER

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The velocity and temperature distributions in a viscous incompressible fluid flow in a two-dimensional diffuser are analyzed. Fully developed flow is considered, i.e., the influence of the entrant section is disregarded. It is assumed that the diffuser walls are maintained at a temperature depending on the polar radius. The dynamic viscosity is considered to be an exponential function of the temperature. The problem is reduced to the solution of a system of ordinary differential equations, which is solved by the method of successive approximations. The convergence of the iterative scheme is proved.

1. Consider the steady flow of a viscous fluid in a two-dimensional divergent channel (diffuser). We assume that the viscosity coefficient and temperature are related by the expression

$$\mu = \mu_0 e^{-\beta T} \tag{1.1}$$

in which μ_0 is the viscosity at zero temperature and β is a parameter depending on the kind of fluid and the temperature interval.

The system of flow equations exclusive of inertial and dissipative terms has the form [1]

$$\begin{aligned}
\Delta\Delta\psi = {} & 4\left(\beta^2 \frac{\partial T}{\partial r}\frac{\partial T}{\partial \varphi} - \beta \frac{\partial^2 T}{\partial r\,\partial\varphi}\right)\left(-\frac{1}{r}\frac{\partial^2\psi}{\partial r\,\partial\varphi} + \frac{1}{r^2}\frac{\partial\psi}{\partial\varphi}\right) - \\
& - \beta\frac{\partial T}{\partial r}\left(-\frac{2}{r}\frac{\partial^3\psi}{\partial\varphi^2\,\partial r} - 2r\frac{\partial^3\psi}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} - \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right) - \\
& - \beta\frac{\partial T}{\partial\varphi}\left(-\frac{2}{r}\frac{\partial^3\psi}{\partial r^2\,\partial\varphi} - \frac{2}{r^3}\frac{\partial^3\psi}{\partial\varphi^3} + \frac{2}{r^2}\frac{\partial\psi}{\partial r\,\partial\varphi} - \frac{4}{r^3}\frac{\partial\psi}{\partial\varphi}\right) + \\
& + \left(\beta^2\frac{\partial T^2}{\partial\varphi} - \beta\frac{\partial^2 T}{\partial\varphi^2}\right)\left(\frac{1}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^3}\frac{\partial^2\psi}{\partial\varphi^2} - \frac{1}{r^2}\frac{\partial\psi}{\partial r}\right) - \\
& - \left(\beta^2\frac{\partial T^2}{\partial r} - \beta\frac{\partial^2 T}{\partial r^2}\right)\left(r\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial\varphi^2} - \frac{\partial\psi}{\partial r}\right) \\
a\Delta T = {} & -\frac{1}{r}\frac{\partial\psi}{\partial\varphi}\frac{\partial T}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\varphi}
\end{aligned} \tag{1.2}$$

where ψ is the stream function and a is the constant coefficient of thermal diffusivity.

We specify the boundary conditions and condition of constant volumetric flow across any cross section of the diffuser as follows:

$$\begin{aligned}
& \frac{\partial\psi}{\partial\varphi} = 0, \quad \frac{\partial\psi}{\partial r} = 0 \quad \left(\varphi = \pm\frac{\alpha}{2}\right) \\
& \frac{1}{r}\frac{\partial\psi}{\partial\varphi} \to 0, \quad \frac{\partial\psi}{\partial r} \to 0 \quad (r \to \infty) \\
& \psi\left(r, \frac{\alpha}{2}\right) = -\frac{Q}{2} \quad (\psi(r, 0) = 0)
\end{aligned} \tag{1.3}$$

Here α is the aperture angle of the diffuser, and Q is the volumetric flow rate of the fluid.

Let the temperature at the diffuser walls be given by the function

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$$T = f(r) = \sum_{k=1}^{j} T_k r^{-k} \tag{1.4}$$

where T_k are known numbers.

It is obvious that conditions (1.4) are inapplicable in the vicinity of the origin ($r = 0$) so that the final solution will be meaningful only in the neighborhood of an infinitely distant point. In other words, the influence of the entrant section and temperature conditions at its boundary is neglected.

The controlling variables and parameters in the problem are r, φ, α, μ_0, Q, β, $f(r)$, a, ρ, and c_p (ρ is the density of the fluid, and c_p is the specific heat at constant pressure), so that the dimensionless stream function ψ/Q must depend on dimensionless combinations of the above-named independent variables and parameters:

$$\psi / Q = F(\varphi, \alpha, \beta f(r), P, R, H) \tag{1.5}$$
$$(P = |Q| / a, \quad R = |Q| \rho / \mu_0, \quad H = \beta\mu_0 Q^2 / a\rho c_p J r^2)$$

where P is the Péclet number, R is the Reynolds number, and H is a parameter characterizing dissipation effects.

If the inertia and dissipation of the flow are small ($R \ll 1$, $H \ll 1$),

$$\psi / Q = F(\beta f(r), \varphi, \alpha, P) \tag{1.6}$$

It follows from (1.6) that $\partial F/\partial \neq 0$ and, therefore, the radial directions in the diffuser are not streamlines, whereas for μ = const ($\beta = 0$) the flow is everywhere radial.

If energy dissipation is taken into account and the wall temperature is assumed to be constant, the variable r enters into the parameter H, and the radiality of the flow is violated. For constant viscous properties it is impossible to form dimensionless criteria containing r in nonzero powers. For μ = const an exact solution of the energy equation has been obtained in [2, 3].

We seek the solution of Eqs. (1.2) in the series form

$$\psi(r, \varphi) = \sum_{k=0}^{\infty} a_k(\varphi) r^{-k}, \qquad T(r, \varphi) = \sum_{k=0}^{\infty} b_k(\varphi) r^{-k} \tag{1.7}$$

Substituting (1.7) into (1.2) and equating coefficients of like powers of r, we obtain the system of ordinary differential equations

$$\begin{aligned}
& a_k^{IV} + (2k^2 + 4k + 4) a_k'' + k^2 (k+2)^2 a_k = \\
& = F_k(a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_k) \\
& a \left[b_k'' + \left\{ k^2 + kP \left(\frac{\cos 2\varphi - \cos\alpha}{\sin\alpha - \alpha\cos\alpha} \right) \right\} b_k \right] = \Psi_k(a_1, \ldots, a_{k-1}, b_0, \ldots, b_{k-1}) \\
& F_k(\varphi) = \beta^2 \sum_{n=0}^{k} \sum_{m=0}^{k} \{ b'_{k-(m+n)} b_n' (m(m+2) a_m - a_m'') - \\
& -(k-(m+n)) b_{k-(m+n)} [4(m+1) b_n' a_m' + n b_n (m(m+2) a_m - a_m'')] \} + \\
& + \beta \sum_{n=0}^{k} \{ (k-n)(2n+3) a_n'' b_{k-n} + 4(k-n)(n+1) b_{k-n}' a_n' + \\
& + (k-n) 2n (n+2)(2n+1) b_{k-n} a_n + (2n^2 + 4n + 4) b_{k-n}' a_n' - \\
& - b_{k-n}'' [n(n+2) a_n - a_n''] + (k-n)(k-n+1) \times \\
& \times b_{k-n} [n(n+2) a_n - a_n''] + 2 b_{k-n} a_n''' \} \\
& \Psi_k(\varphi) = \sum_{n=0}^{k} \{ -n a_n b_{k-n}' + (k-n) a_n' b_{k-n} \}
\end{aligned} \tag{1.8}$$

where $m + n \leq k$.

It was assumed in the derivation of Eqs. (1.8) that

$$a_0(\varphi) = -\frac{Q(\sin 2\varphi - 2\varphi\cos\alpha)}{2(\sin\alpha - \alpha\cos\alpha)}, \quad b_0(\varphi) \equiv 0 \tag{1.9}$$

Equations (1.9) describe a slow isothermal flow of a viscous fluid, representing the zeroth approximation of the original problem [4].

With allowance for (1.7), conditions (1.3) and (1.4) assume the form

$$a_k(\pm \alpha/2) = 0, \quad b_k(\pm \alpha/2) = T_k \quad (k = 1, 2, 3 \ldots)$$
$$a_k'(\pm \alpha/2) = 0, \quad a_0(\alpha/2) = -Q/2 \quad (k = 0, 1, 2 \ldots) \tag{1.10}$$

We have thus reduced the problem to the integration of inhomogeneous linear equations, one of which has constant coefficients, while the other is a Mathieu equation [5].

Using the Lagrange method [6], we write the general solution of the system (1.8) in quadratures:

$$a_k(\varphi) = A_k \sin k\varphi + B_k \sin(k+2)\varphi + \frac{1}{k}\left(-\cos k\varphi \int_0^\varphi V_k \sin k\theta \, d\theta + \sin k\varphi \int_0^\varphi V_k \cos k\theta \, d\theta\right)$$
$$b_k(\varphi) = D_k y_{1k}(\varphi) + \frac{1}{c^2}\left(-y_{1k}\int_0^\varphi \Psi_k y_{2k}\, d\theta + y_{2k}\int_0^\varphi \Psi_k y_{1k}\, d\theta\right) \tag{1.11}$$

Here sin kφ, sin (k+ 2) φ. cos kφ, cos (k+ 2) φ is the fundamental system of solutions of the first of Eqs. (1.8) without the right-hand side, $y_{1k}(\varphi)$, $y_{2k}(\varphi)$ are the even and odd solutions of the Mathieu equation

$$V_k(\varphi) = \frac{1}{k+2}\left(-\cos(k+2)\varphi \int_0^\varphi F_k \sin(k+2)\theta \, d\theta + \sin(k+2)\varphi \int_0^\varphi F_k \cos(k+2)\theta \, d\theta\right)$$

$F_k(\varphi)$, $\Psi_k(\varphi)$ are the right-hand sides of the system (1.8), A_k, B_k, and D_k are constants determined from conditions (1.10), and $c^2 = y_{1k}(0)y_{2k}'(0) - y_{2k}(0)y_{1k}'(0)$ is the fundamental identity.

If the boundary conditions (1.4) are represented in the form

$$T(r, \pm \alpha/2) = T_k r^{-k} \tag{1.12}$$

where k is a large number, corresponding to a rapid variation of the diffuser wall temperature, or if the fluid flow is analyzed for small P, then for the solution of the Mathieu equation we can invoke the asymptotic expressions

$$y_{1k}(\varphi) = \sum_{i=-\infty}^{\infty} (-1)^i I_i(h_k) \cos(m_k^{0.5} - 2i)\varphi$$
$$y_{2k}(\varphi) = \sum_{i=-\infty}^{\infty} (-1)^i I_i(h_k) \sin(m_k^{0.5} - 2i)\varphi$$
$$m_k = k^2 - \frac{kP\cos\alpha}{\sin\alpha - \alpha\cos\alpha}, \quad q_k = -\frac{kP}{\sin\alpha - \alpha\cos\alpha}, \quad h_k = q_k/2m_k^{0.5} \tag{1.13}$$

in which $I_i(h_k)$ are Bessel functions.

The criterion for the applicability of Eqs. (1.13) is the quantity $m_k \gg q_k$. Relation (1.13) is well suited to the calculation of the higher approximations. It follows from the form of the boundary conditions (1.12) that the first nonzero approximation for the second of Eqs. (1.8) is the function

$$b_k(\varphi) = \frac{T_k}{y_{1k}(\alpha/2)} y_{1k}(\varphi) \tag{1.14}$$

To find the function $a_k(\varphi)$ we need to determine the integral of the equation

$$a_k^{IV} + (2k^2 + 4k + 4)a_k'' + k^2(k+2)^2 a_k =$$
$$= \beta\left[(2k - k^2)a_0''b_k + 4(k+1)a_0'b_k' + b_k''a_0'' + 2b_k'a_0''\right] \tag{1.15}$$

Substituting (1.14) and (1.9) into the right-hand side of (1.15), we obtain

$$a_k^{IV} + (2k^2 + 4k + 4) a_k'' + k^2 (k+2)^2 a_k = K \sum_{i=-\infty}^{\infty} (-1)^i I_i(h_k) \times$$
$$\times [A_i \sin(m_k^{1/2} - 2i + 2)\varphi + B_i \sin(m_k^{1/2} - 2i - 2)\varphi +$$
$$+ C_i \sin(m_k^{1/2} - 2i)\varphi] \quad (1.16)$$
$$\left(A_i = 1 - (m_k^{1/2} - 2i - k + 1)^2, \quad B_i = 1 - (m_k^{1/2} - 2i + k - 1)^2,\right.$$
$$\left. C_i = -4(k+1)(m_k^{1/2} - 2i)\cos\alpha, \quad K = \frac{\beta T_k Q}{y_{1k}(\alpha/2)(\sin\alpha - \alpha\cos\alpha)}\right)$$

It is important to note that $I_i(h_k)$ are rapidly decaying functions as $i \to \infty$ and only a few terms need to be retained for specific calculations.

We calculate the general solution of Eq. (1.16) by the method of indeterminate coefficients:

$$a_k(\varphi) = A_k \sin k\varphi + B_k \sin(k+2)\varphi + K \sum_{i=-\infty}^{\infty} (-1)^i I_i(h_k) [\Delta_i^1 \sin(m_k^{1/2} - 2i + 2)\varphi$$
$$+ \Delta_i^2 \sin(m_k^{1/2} - 2i - 2)\varphi + \Delta_i^3 \sin(m_k^{1/2} - 2i)\varphi] \quad (1.17)$$
$$\left(\Delta_i^1 = \frac{A_i}{L(m_k^{1/2} - 2i + 2)}, \quad \Delta_i^2 = \frac{B_i}{L(m_k^{1/2} - 2i - 2)}, \quad \Delta_i^3 = \frac{C_i}{L(m_k^{1/2} - 2i)}\right)$$

where L(p) is the value of the characteristic equation (1.16) at the point p,

$$A_k = -\begin{vmatrix} \Phi(\alpha/2)\sin k\alpha/2 & \sin(k+2)\alpha/2 \\ \Phi'(\alpha/2)k\cos k\alpha/2 & (k+2)\cos(k+2)\alpha/2 \end{vmatrix} t^{-1}$$
$$B_k = -\begin{vmatrix} \sin k\alpha/2 & \Phi(\alpha/2)\sin(k+2)\alpha/2 \\ k\cos k\alpha/2 & \Phi'(\alpha/2)(k+2)\cos(k+2)\alpha/2 \end{vmatrix} t^{-1}$$
$$t = \sin(k+1)\alpha - (k+1)\sin\alpha$$

and $\Phi(\alpha/2)$ and $\Phi'(\alpha/2)$ are the values of the particular solution and its derivative at the diffuser wall.

The problem is not solvable for all values of the diffuser angle; rather it is solvable only for the values determined by the inequalities $\alpha < 2.545$, $y_{1k}(\alpha/2) \neq 0$.

We now determine the structure of the flow. To simplify the calculations we assume that qualitative information about the behavior of a streamline is afforded by the principal term of the expansions of Eq. (1.13) in powers of the parameter P. We assume that the wall temperature is equal to $T_1 r^{-1}$. It must then be assumed in relations (1.16) that

$$k = 1, \quad A_0 = B_0 = 0, \quad C_0 = -8\cos\alpha$$
$$I_0(0) = 1, \quad I_i(0) = 0 \quad (i \geqslant 1), \quad K = \beta T_1 Q \left[\cos\frac{\alpha}{2}(\sin\alpha - \alpha\cos\alpha)\right]^{-1} \quad (1.18)$$

We solve Eq. (1.16), taking (1.9), (1.10), and (1.18) into account:

$$a_1(\varphi) = A_1 \sin\varphi + B_1 \sin 3\varphi + K\cos\alpha\,(\varphi\cos\varphi)/2 \quad (1.19)$$

where the constants A_1 and B_1 are determined from the boundary conditions (1.10):

$$A_1 = K \operatorname{ctg}\alpha \left[\frac{(\sin\alpha - \alpha)(2\cos\alpha + 1) + 2\alpha\sin^2\alpha}{8(\cos\alpha - 1)}\right]$$
$$B_1 = K \operatorname{ctg}\alpha \left[\frac{\alpha - \sin\alpha}{8(\cos\alpha - 1)}\right]$$

If we consider the diffuser angle to be small, the functions (1.9) and (1.19) can be expanded in power series in α and φ, whereupon we obtain for the stream function and the radial velocity component

$$\psi(r,\varphi) = -\frac{3}{2}\frac{Q}{\alpha^3}\left(\alpha^2 - \frac{4}{3}\varphi^2\right)\varphi - \frac{1}{r}\frac{3T_1 Q}{80\alpha^3}(4\varphi^2 - \alpha^2)^2\varphi$$
$$v_r(r,\varphi) = \frac{3Q}{2r\alpha^3}(\alpha^2 - 4\varphi^2) + \frac{1}{r^2}\frac{3T_1 Q}{80\alpha^3}(\alpha^4 - 24\alpha^2\varphi^2 + 80\varphi^4) \quad (1.20)$$

Analyzing the radial velocity distribution, we see that for $T_1 > 0$ (walls colder than the fluid) the velocity increases on the diffuser axis, while in the vicinity of the walls it is lower than for isothermal flow. For $T_1 < 0$ the velocity profile is more uniformly distributed over the diffuser cross section.

Using (1.20) and following Slezkin [7], we represent the stream function in the form

$$\psi(r, \varphi) = -\lambda_1(\varphi) - T_1\lambda_2(\varphi)/r$$
$$\lambda_1(\varphi) = \frac{3}{2}\frac{Q}{\alpha^3}\left(\alpha^2 - \frac{4}{3}\varphi^2\right)\varphi, \quad \lambda_2(\varphi) = \frac{3Q}{80\alpha^3}(4\varphi^2 - \alpha^2)^2\varphi \tag{1.21}$$
$$\lambda_1(\varphi) \geqslant 0, \quad \lambda_2(\varphi) \geqslant 0 \quad \text{for} \quad 0 \leqslant \varphi \leqslant \alpha/2$$

We consider the case $T_1 > 0$. Inasmuch as we are investigating the behavior of the streamlines for large r, it is logical to suppose for divergent flow that $\psi(r, \varphi) < 0$. Equation (1.21) implies

$$r = -T_1\lambda_2(\varphi)/(\psi(r, \varphi) + \lambda_1(\varphi)) \tag{1.22}$$

For the quantity r given by Eq. (1.22) to be positive, the value of φ must lie in the interval $0 \le \varphi \le \varphi_1$, where φ_1 satisfies the equation

$$\psi(r, \varphi_1) + \lambda_1(\varphi_1) = 0 \tag{1.23}$$

The streamlines ψ = const in this case are situated closer to the diffuser axis, being convex relative to it.

If $T_1 < 0$, the quantity φ must vary in the interval $\varphi_1 \le \varphi \le \alpha/2$, and the trajectories of the fluid particles, on entering the diffuser from infinity, become deflected toward the diffuser walls, being concave relative to the line (1.23).

If we assume that the diffuser wall temperature is constant but different at different walls:

$$T(r, +\alpha/2) = T_0, \quad T(r, -\alpha/2) = T_1$$

we obtain the following equation for the radial velocity:

$$z'' - 3\left(\frac{T_0 - T_1}{\alpha}\right)z' + 4z = \exp 3\left(\frac{T_0 - T_1}{\alpha}\right)\varphi \tag{1.24}$$

in which $z(\varphi) = v_r r$. The general solution of (1.24) has the form

$$z(\varphi) = e^{0.5\delta\varphi}(A_1e^{\omega\varphi} + B_1e^{-\omega\varphi}) + 0.25D_1e^{\delta\varphi} \qquad (\delta = 3(T_0 - T_1)/\alpha)$$

where $\omega = (0.25\,\delta^2 - 4)^{1/2}$ is in general a complex quantity.

The constants A_1, B_1, and C_1 are evaluated from the conditions

$$z(\pm\alpha/2) = 0, \quad \int_{-\alpha/2}^{+\alpha/2} z(\varphi)\,d\varphi = Q$$

2. We investigate the convergence of the successive-approximation scheme, using the same method of proof as in [7].

We estimate the moduli of the particular solution of the first equation of the system (1.8) and its first three derivatives. We have from (1.11)

$$f_k(\varphi) = \frac{1}{k}\left(-\cos k\varphi \int_0^\varphi V_k \sin k\theta\,d\theta + \sin k\varphi \int_0^\varphi V_k \cos k\theta\,d\theta\right)$$
$$\left(V_k(\varphi) = \frac{1}{k+2}\left(-\cos(k+2)\varphi \int_0^\varphi F_k \sin(k+2)\theta\,d\theta + \sin(k+2)\varphi \int_0^\varphi F_k \cos(k+2)\theta\,d\theta\right)\right) \tag{2.1}$$

We introduce the notation

$$\max|V_k| \leqslant M_k, \qquad \max|V_k'| \leqslant N_k, \qquad \max|F_k| \leqslant R_k \tag{2.2}$$

The existence of the numbers M_k, N_k, and R_k is inferred from the fact that the investigated functions are continuous on a closed interval. Using relation (2.2) and computing the definite integrals, we obtain the upper bounds

$$|f_k| \leqslant M_k\,|1-\cos k\varphi|\,/\,k^2, \qquad |f_k''| \leqslant M_k\,|\cos k\varphi|$$
$$|f_k'| \leqslant M_k\,|\sin k\varphi|\,/\,k, \qquad |f_k'''| \leqslant M_k k\,|\sin k\varphi|\,k + N_k \tag{2.3}$$

Analogously, we have

$$|V_k| \leqslant R_k|1-\cos(k+2)\varphi|\,/\,(k+2)^2, \ |V_k'| \leqslant R_k|\sin(k+2)\varphi|\,/\,(k+2) \tag{2.4}$$

Clearly, we can use the following expressions in the role of M_k and N_k:

$$M_k = 2R_k\,/\,(k+2)^2, \qquad N_k = R_k\,/\,(k+2) \tag{2.5}$$

Consequently, taking (2.5) into account, we obtain the final inequalities for the particular solution and its first three derivatives:

$$|f_k| \leqslant 2R_k\,|1-\cos k\varphi|\,/\,k^2(k+2)^2$$
$$|f_k'| \leqslant 2R_k\,|\sin k\varphi|\,/\,k(k+2)^2$$
$$|f_k''| \leqslant 2R_k\,|\cos k\varphi|\,/\,(k+2)^2 \tag{2.6}$$
$$|f_k'''| \leqslant R_k\,[2k\,|\sin k\varphi|\,/\,(k+2)^2 + 1\,/\,(k+2)]$$

Next we estimate the complete solution and its derivatives. Taking Conditions (1.10) into account, we obtain for the stream function in the k-th approximation

$$a_k(\varphi) = f_k(\varphi) + f_k(\alpha/2)\chi_{1k}(\alpha,\varphi) + f_k'(\alpha/2)\chi_{2k}(\alpha,\varphi)$$
$$\chi_{1k} = \begin{vmatrix} \sin k\varphi & \sin(k+2)\varphi \\ k\cos(k\alpha/2) & (k+2)\cos[(k+2)\alpha/2] \end{vmatrix} t^{-1} \tag{2.7}$$
$$\chi_{2k} = \begin{vmatrix} \sin(k+2)\varphi & \sin k\varphi \\ \sin[(k+2)\alpha/2] & \sin(k\alpha/2) \end{vmatrix} t^{-1}$$

We introduce the function

$$|\varphi_k| = 2\,|1-\cos k\varphi| + 2\,|1-\cos(k\alpha/2)| \times \chi_{1k}(\alpha,\varphi) + 2k\,|\sin(k\alpha/2)|\,\chi_{2k}(\alpha,\varphi) \tag{2.8}$$

With the help of (2.8) we obtain the following inequalities for the upper bounds of the complete solution and its first three derivatives

$$|a_k| \leqslant R_k\,|\varphi_k|\,/\,k^2(k+2)^2, \quad |a_k''| \leqslant R_k\,|\varphi_k''|\,/\,k^2(k+2)^2$$
$$|a_k'| \leqslant R_k|\varphi_k'|\,/\,k^2(k+2)^2, \quad |a_k'''| \leqslant R_k\,[\,|\varphi_k'''|\,/\,k^2 + k + 2]\,/\,(k+2)^2 \tag{2.9}$$

Using (2.9), we obtain

$$k(k+2)\,|a_k| \leqslant R_k\,|\varphi_k|\,/\,k(k+2)$$
$$4(k+1)\,|a_k'| \leqslant 4(k+1)\,R_k\,|\varphi'_k|\,/\,k^2(k+2)^2$$
$$(2k+3)\,|a_k''| \leqslant (2k+3)\,R_k\,|\varphi_k''|\,/\,k^2(k+2)^2 \tag{2.10}$$
$$2k(k+2)(2k+1)\,|a_k| \leqslant 2(2k+1)\,R_k\,|\varphi_k|/k(k+2)$$
$$(2k^2+4k+4)/\,|a_k'| \leqslant (2k^2+4k+4)\,R_k\,|\varphi_k'|/k^2(k+1)^2$$

The factors multiplying R_k on the right-hand sides of inequalities (2.9) and (2.10) are bounded functions for k = 1, 2, 3,.... Denoting by h the maximum number bounding those functions, we have

$$|a_k'| \leqslant U_k, \ |a_k''| \leqslant U_k, \ |a_k'| \leqslant U_k \quad |a_k'''| \leqslant U_k, \ U_k = hR_k \tag{2.11}$$

The left-hand sides of inequalities (2.10) are also less than the number U_k introduced above.

Next we estimate the solution of the second equation of the system (1.8). We have on the basis of the boundary conditions (1.10)

$$b_k(\varphi) = \frac{T_k - f_{1k}(\alpha/2)}{y_{1k}(\alpha/2)}\,y_{1k}(\varphi) + f_{1k}(\varphi)$$

$$\left(f_{1k}(\varphi) = \frac{1}{c^2}\left(-y_{1k}\int_0^\varphi \Psi_k y_{2k}d\theta + y_{2k}\int_0^\varphi \Psi_k y_{1k}\,d\theta\right)\right) \tag{2.12}$$

Making use of the asymptotic expressions (1.13), we readily verify that c^2 and k attain the same order of magnitude as $k \to \infty$. We then let

$$\max|\Psi_k(\varphi)| \leqslant H_k$$

Now

$$\begin{aligned}
|f_{1k}| &\leqslant \frac{H_k}{ac^2}\left|-y_{1k}\int_0^\varphi y_{2k}d\theta + y_{2k}\int_0^\varphi y_{2k}d\theta\right| \\
|f_{2k}'| &\leqslant \frac{H_k}{ac^2}\left|-y_{1k}'\int_0^\varphi y_{2k}d\theta + y_{2k}'\int_0^\varphi y_{1k}d\theta\right| \\
|f_{1k}''| &\leqslant \frac{H_k}{ac^2}\left|-y_{1k}''\int_0^\varphi y_{2k}d\theta + y_{2k}''\int_0^\varphi y_{1k}d\theta + c^2\right|
\end{aligned} \tag{2.13}$$

Taking inequalities (2.13) into account, we obtain the following bounds for the complete solution of the second equation (1.8):

$$\begin{aligned}
|b_k| &\leqslant |T_k|\,|g_{1k}(\varphi)| + \frac{H_k}{ac^2}\left(|g_{1k}(\varphi)|\left|g_{2k}\frac{\alpha}{2}\right| + |g_{2k}(\varphi)|\right) \\
|b_k'| &\leqslant |T_k|\,|g_{1k}'(\varphi)| + \frac{H_k}{ac^2}\left(|g_{1k}'(\varphi)|\left|g_{2k}\frac{\alpha}{2}\right| + |g_{2k}'(\varphi)|\right) \\
|b_k''| &\leqslant |T_k|\,|g_{1k}''(\varphi)| + \frac{H_k}{ac^2}\left(|g_{1k}''(\varphi)|\left|g_{2k}\frac{\alpha}{2}\right| + |g_{2k}''(\varphi)|\right) \\
&\left(g_{1k} = \frac{y_{1k}(\varphi)}{y_{1k}(\alpha/2)}, \quad g_{2k}(\varphi) = -y_{1k}\int_0^\varphi y_{2k}d\theta + y_{2k}\int_0^\varphi y_{1k}d\theta\right)
\end{aligned} \tag{2.14}$$

According to inequalities (2.14), we have

$$k(k+1)|b_k(\varphi)| \leqslant k(k+1)|T_k|\,|g_{1k}(\varphi)| + \frac{k(k+1)H_k}{ac^2}(|g_{1k}|\,|g_{2k}(\alpha/2)| + |g_{2k}'|)$$

The factors multiplying $|T_k|$ in inequalities (2.14) are bounded functions, because k assumes finitely many values. The factors multiplying H_k are also bounded for all values of k, so that, denoting by d a number greater than the maximum of the indicated functions, we arrive at the inequalities

$$\begin{aligned}
&k(k+1)\,|b_k| \leqslant (|T_k| + H_k/a)\,d = \theta_k \\
&|b_k'| \leqslant \theta_k, \qquad |b_k''| \leqslant \theta_k
\end{aligned} \tag{2.15}$$

We form the series

$$U = \sum_{k=0}^\infty U_k s^k, \quad \theta = \sum_{k=0}^\infty \theta_k s^k \quad (s = 1/r) \tag{2.16}$$

If these series converge in the vicinity of the zero point, the series (1.7) will converge uniformly by virtue of inequalities (2.11) and (2.15). The following holds for the general terms of the series (2.16):

$$\begin{aligned}
U_k &= 5\beta^2 h \sum_{n=0}^k \sum_{m=0}^k U_m \theta_{k-(m+n)}\theta_m + 9\beta h \sum_{n=0}^k U_n \theta_{k-n} \\
\theta_k &= \frac{2d}{a}\sum_{n=1}^k U_n \theta_{k-n} + d\,|T_k|
\end{aligned} \tag{2.17}$$

Equations (2.7) are readily deduced by estimating the moduli of the functions F_k and Ψ_k, determining the numbers R_k and H_k, and taking inequalities (2.11) and (2.15) into account at the same time.

Replacing the general term of the series (2.16) by expressions (2.17), we obtain the relations

$$U = U_0 + 5\beta^2 h U \theta^2 + 9\beta h U \theta, \quad \theta = 2da^{-1}(U - U_0)\theta + dT(s)$$

Here $T(s)$ is a rational function of s by condition (1.4).

Denoting by $W = U\theta$ the product of the series (2.16) and carrying out elementary transformations, we obtain

$$\frac{20\beta^2 hm^2}{a^2} W^3 + \left[\frac{20\beta^2 hm^2}{a} T(s) + \frac{18\beta hm}{a}\right] W^2 + \Big(5\beta^2 hm^2 T^2(s) + 9\beta hm T(s) + \frac{2mU_0}{a} - 1\Big) W + mU_0 T(s) = 0 \quad (m = da(a + 2dU_0)^{-1}) \tag{2.18}$$

It follows from Eq. (2.18) that the function W(s) is algebraic and therefore has a nonzero radius of convergence in the vicinity of s = 0. The radius of convergence is determined by the distance to the nearest singular point of the function W(s). The convergence of W(s) implies the convergence of the series (2.16).

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